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Quantum Fluctuations in a ϕ^4 Field Theory I - the Stability of the Vacuum

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ABSTRACT

We study the effect of quantum fluctuations in a ϕ^4 field theory using a Hartree-type approximation. We reduce the operator field equations into a set of coupled c-number (infinite in number) equations. We show that these equations can also be derived from a variational principle. We carry out the renormalization procedure in the one dimensional theory in detail and demonstrate how the renormalization can be done in a fashion consistent with the Hartree approximation. We then apply the technique to study the effective potential and the stability of the vacuum. We find that the abnormal vacuum is unstable as the coupling becomes strong, and a transition between the abnormal ($<\phi> \neq 0$) and the normal ($<\phi> = 0$) vacua occurs in both the one and the three dimensional theory. A generalization of the theory to include internal symmetries is briefly outlined.

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I. INTRODUCTION

Recently, several groups have studied the nature of relativistic field theories and the possible structure of hadrons using nonperturbative methods.[1-7] These studies are important in view of the fact that the usual perturbation theory failed to supply useful guides in hadron physics. In the MIT bag model,[1] hadrons are described as fundamental constituents (e.g., quarks) trapped inside a cavity, known as a bag. In this model, the bag is put in by hand. In the SLAC model,[2] the hadrons are bound states of fundamental fermion fields (quarks) interacting with a scalar σ field. The expectation value of σ changes in the region of a hadron and produces a bag-like (more precisely, a shell-like) solution in the strong coupling limit. At the moment, their calculation is mainly classical. The effect of quantum fluctuations of both the σ-field and the fermion fields have not been taken into account in computing the bag solution. It is important to know if the basic structure of the solutions is modified, or if the $\langle\sigma\rangle \neq 0$ ground state in which the bags are formed is stable in the strong coupling limit. We shall supply partial answers to these questions.

In this and the subsequent papers, we investigate the effect of the quantum fluctuations in a self-interacting scalar field. We choose to study the ϕ^4 field theory which is the simplest nontrivial self-interacting field theory. To study the quantum fluctuations, we make use of a technique which is a generalization of the self-consistent Hartree approximation.[8]

Aside from the problem of renormalization, the Hartree approximation is well-developed in nonrelativistic manybody physics. To test the accuracy of the approximation and to introduce the mathematics, we first apply our method to an anharmonic oscillator whose numerical solutions are known. We find to our surprise that the self-consistent Hartree approximation leads to a ground state energy which is within 2% of the exact answer for the full range of the coupling constant. This indicates that our approximation may have a validity even in the strong coupling limit.

We then apply the Hartree method to the ϕ^4 theory and reduce the operator field equations into an infinite set of coupled c-number equations. The quantum fluctuations and various wave functions can be determined self-consistently. The problem can also be formulated as a variational principle.

In the remainder of this paper, we concentrate on the stability of the ground state. Without the quantum fluctuations, the vacuum state can be determined trivially by minimizing the classical Hamiltonian. With quantum fluctuations, the situation is more complicated. The quantum fluctuation leads to a divergent effective potential. Ths divergence must be removed by renormalizations. Fortunately, in the field theory of one space and one time dimension, the renormalization procedure developed by Dashen, Hasslacher, and Neveu[5] can be applied to our problem. We demonstrate in detail how this can be done in a fashion consistent with the

Hartree approximation. A possible generalization of the renormalization procedure with the Hartree approximation to three space and one time dimension is outlined, but its self-consistency has not been verified. Using these renormalization procedures, we are able to study the energy density associated with various ground states. We show that, as the coupling becomes strong enough, a transition between the abnormal (i.e., $<\phi> = c \neq 0$) and the normal ($<\phi> = 0$) vacua occurs in both the one dimensional and the three dimensional theory.

In subsequent papers,[9] we will study the effect of quantum fluctuation on the one-particle state. We will demonstrate in the one-mode approximation that the one-particle state does create a bag-like configuration through self-interaction. The size of the bag increases as the coupling strength becomes stronger. The translational and Galilean invariance of these bag-like solutions will be demonstrated.

II. A SIMPLE EXAMPLE

In this paper, we shall apply a generalized Hartree approximation to relativistic field theory. To illustrate the method, we apply the approximation to a quantum harmonic oscillator which represents the simplest ϕ^4 type interaction and whose numerical solution is known.[10] Following the notation of Bender and Wu,[10] the Hamiltonian of an anharmonic oscillator can be written as

$$H = p^2 + \tfrac{1}{4}x^2 + \frac{\lambda}{4}x^4 \tag{2.1}$$

with the usual commutator relation

$$[p, x] = i \quad . \tag{2.2}$$

The unperturbed part of H in (2.1) describes a particle of mass $\mu = \frac{1}{2}$ moving in a harmonic well with frequency $\omega_0=1$. Heisenberg's equation of motion for this anharmonic system is

$$\ddot{x} + x + 2\lambda x^3 = 0 \quad . \tag{2.3}$$

It is the nonlinear nature of the λx^3 term in (2.3) which makes the theory nonsolvable analytically. To give an approximate solution to (2.1) or (2.3), we approximate the interaction term in (2.1) by[11]

$$x^4 \to 6\langle x^2\rangle x^2 - 3\langle x^2\rangle^2 \quad , \tag{2.4}$$

or equivalently the x^3-term in (2.3) by

$$x^3 \to 3\langle x^2\rangle x \quad . \tag{2.5}$$

This is a Hartree-type approximation. In (2.4), (2.5), we have kept the lowest order quantum fluctuations in the corresponding terms. Note that $\langle x\rangle = 0$ for all energy eigenstates, and that $\langle x^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ is indeed the quantum fluctuation. Under this approximation, we can reduce the anharmonic oscillator into an harmonic oscillator,

$$\begin{aligned} H &= p^2 + (\tfrac{1}{4} + \tfrac{3\lambda}{2}\langle x^2\rangle)x^2 - \tfrac{3\lambda}{4}\langle x^2\rangle^2 \\ &= p^2 + \tfrac{1}{4}\omega^2 x^2 - \tfrac{3\lambda}{4}\langle x^2\rangle^2 \end{aligned} \tag{2.6}$$

with frequency

$$\omega^2 = 1 + 6\lambda\langle x^2\rangle \quad . \tag{2.7}$$

From (2.6), we can in turn compute the expectation value of x^2. Using the virial theorem, we have

$$\tfrac{1}{4}\omega_n^2 \langle x^2 \rangle_n = \tfrac{1}{2}(n+\tfrac{1}{2})\omega_n \quad ,$$

or

$$\langle x^2 \rangle_n = \frac{2n+1}{\omega_n} \tag{2.8}$$

for the n^{th} eigenstate. Now, we impose the self-consistency requirement that $\langle x^2 \rangle$ introduced in (2.4) (or in (2.5)) is the same $\langle x^2 \rangle_n$ as obtained in (2.8). From (2.7) and (2.8), we can solve for $\langle x^2 \rangle_n$ and ω_n self-consistently. In particular, $\langle x^2 \rangle_n$ obeys a cubic equation

$$6\lambda \langle x^2 \rangle_n^3 + \langle x^2 \rangle_n^2 - (2n+1)^2 = 0 \quad . \tag{2.9}$$

Once $\langle x^2 \rangle_n$ and ω_n are known, we can compute the n^{th} energy eigenvalue as

$$E_n = (n+\tfrac{1}{2})\omega_n - \frac{3\lambda}{4}\langle x^2 \rangle_n^2 \quad . \tag{2.10}$$

The first term in (2.10) can be interpreted as the n^{th} energy eigenvalue for the induced harmonic oscillator. At first sight, the second term, $-\frac{3\lambda}{4}\langle x^2 \rangle^2$, is puzzling. It appears to imply a negative contribution from the interaction term $H_I = \frac{\lambda}{4}x^4$. After a little work we can demonstrate that this is not true because ω_n also contains a λ-dependence. In fact, the ω-term in (2.10) always increases with λ fast enough to make the total contribution of H_I positive. To see it explicitly, we compute the ω_n and E_n to order $O(\lambda)$ explicitly,

$$\omega_n = 1 + 3(2n+1)\lambda + O(\lambda^2) \quad , \tag{2.11}$$

and

$$E_n = (n+\tfrac{1}{2}) + \frac{3}{4}\lambda(2n+1)^2 + O(\lambda^2) \tag{2.12a}$$

$$= (n+\tfrac{1}{2})\omega - \frac{3\lambda}{4}(2n+1)^2 + O(\lambda^2) \quad . \tag{2.12b}$$

Indeed, the additional contribution to E_n is positive if we express the result in terms of the unperturbed frequency $\omega_0 = 1$. It appears to be negative only if we express our result in terms of the perturbed frequency ω. This is a rather trivial point, but it will emerge again in the field theory calculation.

To estimate the accuracy of our approximation, we compare our results with the exact numerical calculation carried out by Schwartz and Simon.[10] For the ground state energy E_0, our result agrees with the exact numerical result to within 2% for the full range of λ between 0 and ∞.[12] In particular, as $\lambda \to \infty$, our result predicts

$$E_0(\lambda) = \frac{3}{8}\cdot(6\lambda)^{\frac{1}{3}} = 0.68142\ \lambda^{\frac{1}{3}}$$

while the exact answer is

$$\left[E_0(\lambda)\right]_{\text{exact}} = 0.66799\ \lambda^{\frac{1}{3}} \quad .$$

The agreement is even better for smaller λ, and our result gives the exact answer as $\lambda \to 0$. The agreement is less impressive for the higher excited states. However, even in these cases, our method gives the correct n and λ dependence with a coefficient which is always within

20% of the true value.

In the next section, we shall generalize our method to the field theory calculation. Judging from the excellent agreement in the anharmonic oscillator calculation, we anticipate that our approximation should give a qualitatively correct picture for the ground state as well as for states with a small number of particles (i.e., with small occupation number).

III. THE MODEL

The theory that we study in this paper is a self-interacting ϕ^4 model

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{4}(\phi^2 - c^2)^2 \quad . \tag{3.1}$$

We restrict ourselves to $c^2 > 0$, $g > 0$ even though our method is applicable to $c^2 < 0$ as well. The Hamiltonian density associated with $\mathscr{L}$ in (3.1) is

$$\mathscr{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \frac{g}{4}(\phi^2 - c^2)^2 \quad . \tag{3.2}$$

It is easy to see that, as a classical system, the ground state of (3.2) is located at $\phi^2 = c^2$. A system in which the ground state is associated with a particular value of ϕ, say $\phi = c$, is known to have a spontaneously broken symmetry. (Here, the broken symmetry is $\phi \to -\phi$.) We shall refer to the quantum mechanical analog of this ground state as an abnormal vacuum while the ground state associated with $\phi = 0$ will be referred to as the normal vacuum. We shall investigate the effect of quantum fluctuations on the stability of this abnormal vacuum. For simplicity, we shall start with a real field ϕ , and ignore temporarily the need for renormalization. The introduction of internal symmetries, and renormalizations will be discussed later.

A. Field Equations

From (3.1), we obtain the field equation

$$\partial^2\phi + g(\phi^2 - c^2)\phi = 0 \quad . \tag{3.3}$$

Unlike the x in an anharmonic oscillator, ϕ usually has a nonvanishing expectation value. We can separate ϕ into a c-number part ϕ_c and an operator part ϕ_q through

$$\phi = \phi_c + \phi_q \tag{3.4}$$

with

$$\phi_c = \langle \phi \rangle \tag{3.5}$$

and

$$\langle \phi_q \rangle = 0 \quad . \tag{3.6}$$

Obviously, the separation (3.4) is not unique. It depends on the particular reference state $| \, \rangle$ with which we compute the expectation value. The reference state may be a vacuum state, a one-particle state, etc. The choice of the reference state is based on the particular problem that we are investigating.

In order to solve (3.3), we make the Hartree type approximation

$$\phi^3 \rightarrow 3\langle \phi^2 \rangle \phi - 2\langle \phi \rangle^3 \quad . \tag{3.7}$$

In (3.7), an extra term $2<\phi>^3$ is subtracted to account for $<\phi>\neq 0$. In terms of ϕ_c and ϕ_q, (3.7) is equivalent to two relations:

$$\phi_q^2 \to <\phi_q^2> \quad , \tag{3.8}$$

$$\phi_q^3 \to <\phi_q^2>\phi_q \quad . \tag{3.9}$$

Indeed, (3.8) and (3.9) include the lowest order quantum fluctuations characterizing the Hartree approximation. A similar approximation can be made for ϕ_q^4 as mentioned in the previous section.

Under these approximations, we have

$$\partial^2\phi + g(3<\phi^2> - c^2)\phi - 2g<\phi>^3 = 0 \quad . \tag{3.10}$$

Equation (3.10) can be separated into two equations

$$\partial^2\phi_c + g(\phi_c^2 - c^2)\phi_c + 3g<\phi_q^2>\phi_c = 0 \quad , \tag{3.11}$$

$$\partial^2\phi_q + g(3\phi_c^2 - c^2)\phi_q + 3g<\phi_q^2>\phi_q = 0 \quad . \tag{3.12}$$

Equation (3.11) is a c-number equation, and (3.12) is linear in the field operator ϕ_q. It is now possible to expand ϕ_q as combinations of creation and annihilation operators

$$\phi_q = \sum_n (\psi_n^*(x)\, e^{i\omega_n t}\, a_n^\dagger + \psi_n(x)\, e^{-i\omega_n t}\, a_n) \tag{3.13}$$

where the c-number wave functions $\psi_n(\psi_n^*)$ form a complete set of eigenstates, obeying

$$-\omega_n^2\psi_n - \nabla^2\psi_n + g(3\phi_c^2 - c^2 + 3<\phi_q^2>)\psi_n = 0 \tag{3.14}$$

with positive eigen energy ω_n. The creation and annihilation operators are time independent, and satisfy the usual commutator relations

$$\left[a, a\right] = \left[a^{\dagger}, a^{\dagger}\right] = 0 \quad , \tag{3.15}$$

$$\left[a_n, a^{\dagger}_{n'}\right] = \delta_{nn'} \quad . \tag{3.16}$$

The wave functions are normalized by

$$2\omega_n \int dx\, \psi_{n'}^{*}(x)\, \psi_n(x) = \delta_{n'n} \quad . \tag{3.17}$$

Equations (3.11) - (3.17) are valid for a given separation $\phi = \phi_c + \phi_q$. It is important to note that ϕ_c, ϕ_q, $\psi_n(\psi_n^{*})$, and $a_n(a_n^{\dagger})$ all depend on the particular choice of the reference state from which we have made the separation. Wave functions associated with different reference states are usually not orthogonal to each other.

The numerical value of the matrix element $<\phi_q^2>$, of course, does depends on the particular reference state. For a vacuum reference state, we have

$$\phi_c = <\phi> = \text{constant} \quad , \tag{3.18}$$

$$a_n | > = 0 \quad , \qquad \text{all } n \quad , \tag{3.19}$$

and

$$<\phi_q^2(x)> = \sum_n \psi_n^{*}(x)\psi_n(x) \quad . \tag{3.20}$$

We always assume that the operators ϕ_q in $\phi_q(x)^2$ are multiplied symmetrically. Since we shall compare the energy of states associated with different vacua, the concept of normal products is not unique nor useful here. For an arbitrary n-particle reference state with occupation number $(N_1, N_2, \cdots,)$, we have

$$<\phi_q^2> = 2\sum (N_n+\tfrac{1}{2})\psi_n^* \psi_n \quad . \tag{3.21}$$

Given the reference state, Eqs. (3.11), (3.14), and (3.21) lead to a set of coupled c-number equations. These equations can in principle be solved. In practice, however, this is a formidable problem. In the next section, we shall formulate our problem in terms of a variational principle. It is sometimes possible to find the solution by guessing an appropriate trial function.

For the vacuum reference state, it is convenient to express $<\phi_q^2>$ as

$$\begin{aligned} <\phi_q^2(x)> &= \lim_{x'\to x} T<\phi_q(x)\phi_q(x')> \\ &= \lim_{x'\to x} i\, G_F(x-x') \end{aligned} \tag{3.22}$$

where G_F is the causal Green's function defined by

$$\left[\partial_x^2 + g(3\phi_c^2 - c^2 + 3<\phi_q^2>)\right] G_F(x-x') = -\delta^4(x-x') \tag{3.23}$$

and a similar equation on x'. Equations (3.22) and (3.23) lead to a conceptually more transparent, and also a simpler way to determine $<\phi_q^2>$ self-consistently.

B. Hamiltonian

Under the splitting $\phi=\phi_c+\phi_q$, we can decompose the Hamiltonian H into three parts according to

$$H = H_c + H_{cq} + H_q \tag{3.24}$$

where

$$H_c = \int dx \left[\tfrac{1}{2} (\dot{\phi}_c)^2 + \tfrac{1}{2} (\nabla \phi_c)^2 + \frac{g}{4} (\phi_c{}^2 - c^2)^2 \right] \tag{3.25a}$$

$$H_{cq} = \int dx \left[\dot{\phi}_c \dot{\phi}_q + \vec{\nabla} \phi_c \cdot \vec{\nabla} \phi_q + g(\phi_c{}^2 - c^2) \phi_c \phi_q + g\, \phi_c \phi_q{}^3 \right] \tag{3.25b}$$

and

$$H_q = \int dx \left[\tfrac{1}{2} (\dot{\phi}_q)^2 + \tfrac{1}{2} (\vec{\nabla} \phi_q)^2 + \frac{g}{2} (3\phi_c{}^2 - c^2) \phi_q{}^2 + \tfrac{1}{4}\, g\, \phi_q{}^4 \right] . \tag{3.25c}$$

In the decomposition (3.24), H_c is the Hamiltonian for the classical field ϕ_c, H_{cq} is odd in ϕ_q, and H_q is even in ϕ_q. In order to reproduce the coupled equations (3.11) and (3.12), we approximate $\phi_q{}^3$ by Eq. (3.9) and $\phi_q{}^4$ by

$$\phi_q{}^4 \to 6 \langle \phi_q{}^2 \rangle \phi_q{}^2 - 3 \langle \phi_q{}^2 \rangle^2 \tag{3.26}$$

as in Eq. (2.4). Under these approximations and with the help of (3.11), (3.12); we have

$$\begin{aligned} H_{cq} &= \int dx \Big[\dot{\phi}_c \dot{\phi}_q + \nabla(\phi_q \nabla \phi_c) \\ &\qquad + (-\nabla^2 \phi_c + g(\phi_c{}^2 - c^2)\phi_c + 3g \langle \phi_q{}^2 \rangle \phi_c)\phi_q \Big] \\ &= \int dx (\dot{\phi}_c \dot{\phi}_q - \ddot{\phi}_c \phi_q) \quad , \end{aligned} \tag{3.27}$$

and

$$H_q = \sum_n (a_n{}^\dagger a_n + \tfrac{1}{2}) \omega_n - \frac{3}{4} \int dx\, g \langle \phi_q{}^2 \rangle^2 \quad . \tag{3.28}$$

The existence of the cross term H_{cq} is expected by the consistency of our quantization scheme. With this H_{cq} term, we can show in a straightforward fashion that Heisenberg's equations of motion are valid for both the original Hamiltonian with variable ϕ, and the new Hamiltonian with variable ϕ_q:

$$i[H, \phi] = \dot{\phi} \quad , \tag{3.29a}$$

$$i[H, \dot{\phi}] = \ddot{\phi} \quad , \tag{3.29b}$$

and

$$i[H_q, \phi_q] = \dot{\phi}_q \quad , \tag{3.30a}$$

$$i[H_q, \dot{\phi}_q] = \ddot{\phi}_q \quad . \tag{3.30b}$$

H_{cq} serves as the generator for the canonical transformation $\phi \to \phi_q$.

Now, we can compute the energy associated with the reference state. It is given by

$$\begin{aligned} E &= \langle | H | \rangle \\ &= \int dx \left[\tfrac{1}{2}(\dot{\phi}_c)^2 + \tfrac{1}{2}(\nabla\phi_c)^2 + \tfrac{g}{4}(\phi_c{}^2 - c^2)^2 \right] \\ &\quad + \sum_n (N_n + \tfrac{1}{2})\omega_n - \tfrac{3}{4} g \int dx \langle | \phi_q^2 | \rangle^2 \quad . \end{aligned} \tag{3.31}$$

Note that H_{cq} does not contribute to the energy E because it is linear in the creation and annihilation operators. N_n is the occupation number associated with $a_n^\dagger a_n$. All three terms in (3.31) have a simple interpretation. The first term is obviously the energy due to the classical field ϕ_c. The second term, $\sum_n (N_n + \frac{1}{2})\omega_n$, represents the quantum energy due to all the normal modes, including the zero point energy. If the reference state is a vacuum; then, all $N_n=0$ and we have the familiar result for the zero point energy $\frac{1}{2}\sum \omega_n$. As we have demonstrated for the anharmonic oscillator, the last term arises from the nonharmonic nature of the mode oscillations, and is a general feature of the ϕ^4 type interaction system.

C. Variational Principle

Even though we have reduced our quantum mechanical problem into a set of c-number equations; it is still too complicated to solve these equations in general. It is usually much simpler to solve for ω_n and ψ_n with a given ϕ_c. We can then obtain from (3.14), (3.21) and (3.31) the quantities ω_n, ψ_n, and E as functionals of ϕ_c. Of course, the question remains as to what determines the correct choice of ϕ_c. Here, we shall answer this question by finding the correct ϕ_c as a solution to a variational problem. The following variational principle holds:

<u>Lemma</u> A time-independent solution $\phi_c(x)$ for a given reference state can be obtained by minimizing the total energy E(as given in (3.31)) associated with this reference state.

In other words, the requirement

$$\frac{\delta E}{\delta \phi_c} = 0 \tag{3.32}$$

will reproduce the field equation (3.11) for ϕ_c. When there are more than one solutions, the minimization condition is replaced by the stationary condition also specified by (3.32). The proof of this variational principle is straightforward. We leave it as an appendix.

A time-dependent solution ϕ_c can be obtained from a time-independent solution by a Lorentz transformation. By the superposition of these time-dependent states, we can construct solutions having given total momentum. The formulation and physical interpretation of these states will be discussed in a separate paper.[9]

IV. GROUND STATE IN ONE SPACE AND ONE TIME DIMENSION

In the classical limit and when $c^2 > 0$, we know that the ground state is given by $\phi^2 = c^2$. In this section, we shall investigate the effect of quantum fluctuation on the stability of the ground state. We shall compute the energy difference $\Delta E(\phi_c)$ between a state with an arbitrary but constant ϕ_c and "the vacuum" given by $\phi_c = c$. As we shall see, if we ignore the contribution due to the ϕ_q^4 term in our calculation, the energy density difference (ΔE/volume) reduces to the well-known one-loop effective potential $V(\phi_c)$.[13] With the inclusion of the ϕ_q^4 term, our result is qualitatively different from and is physically more interesting than the one-loop calculation. To get some physical insight without involving too much mathematics, we start with the theory in one space and one time dimension.

A. Renormalization

We first write down the ground state energy difference between a reference state with a given $\phi_c(x)$ and the vacuum state with $\phi_c=c$ as

$$\begin{aligned}\Delta E\ (\text{unrenormalized}) &\equiv E(\phi_c) - E(c) \\ &= \int dx\left[\tfrac{1}{2}(\dot{\phi}_c)^2 + \tfrac{1}{2}\left(\frac{\partial\phi_c}{\partial x}\right)^2 + \frac{g}{4}(\phi_c{}^2 - c^2)^2\right] \\ &\quad + \sum_n \tfrac{1}{2}\omega_n(\phi_c) - \frac{3}{4}g\int dx\left(\Sigma\,|\psi_n(\phi_c)|^2\right)^2 \\ &\quad - \sum_n \tfrac{1}{2}\omega_n(c) + \frac{3}{4}g\int dx\left(\Sigma\,|\psi_n(c)|^2\right)^2 .\end{aligned} \tag{4.1}$$

In (4.1), $\omega_n(\phi_c)$ and $\psi_n(\phi_c)$ stand for the n^{th} eigen frequency and wave function associated with ϕ_c. This is the unrenormalized energy difference. It is easy to see that ΔE in (4.1) diverges logarithmically as high frequency modes are included. This is the same kind of divergence as that first studied by Dashen, et al.[5] in a semiclassical calculation. They have demonstrated that this kind of divergence can be removed by a mass, or equivalently, a c-renormalization. It turns out that a similar renormalization procedure is also valid in our model in spite of the presence of a $\phi_q{}^4$ quantum fluctuation term. A sensible energy difference can be obtained after the c (or mass) renormalization.

The renormalized Lagrange function can be written as

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{g}{4}(\phi^2 - c^2)^2 - \tfrac{1}{2}B\phi^2 \tag{4.2}$$

where $\frac{1}{2}B\phi^2$ is the mass counter term. The field Eq. (3.3) is modified to

$$\partial^2\phi + g(\phi^2 - c^2)\,\phi + B\phi = 0 \quad . \tag{4.3}$$

After separating ϕ into ϕ_c and ϕ_q, and under similar approximations to those discussed above, we have

$$\partial^2\phi_c + g(\phi_c{}^2 - c^2)\,\phi_c + (3g\langle\phi_q{}^2\rangle + B)\,\phi_c = 0 \tag{4.4}$$

and

$$\partial^2\phi_q + g(3\phi_c^2 - c^2)\,\phi_q + (3g\langle\phi_q{}^2\rangle + B)\phi_q = 0 \quad . \tag{4.5}$$

The constant B is determined by requiring that $\phi_c = c$ is a static solution (the "abnormal vacuum state") of (4.4). This leads to

$$B = -3g\langle\phi_q{}^2\rangle_{\phi_c = c} = -3g\sum_n |\psi_n(c)|^2 \quad . \tag{4.6}$$

For a general $\phi_c(x) \neq c$, we have

$$\partial^2\phi_c + g(\phi_c{}^2 - c^2 + 3\Delta\langle\phi_q{}^2\rangle)\phi_c = 0 \quad , \tag{4.7}$$

and

$$\partial^2\phi_q + g(3\phi_c{}^2 - c^2 + 3\Delta\langle\phi_q{}^2\rangle)\phi_q = 0 \tag{4.8}$$

where

$$\begin{aligned} \Delta\langle\phi_q{}^2\rangle &\equiv \langle\phi_q{}^2\rangle_{\phi_c} - \langle\phi_q{}^2\rangle_{\phi_c = c} \\ &= \sum_n |\psi_n(\phi_c)|^2 - \sum_n |\psi_n(c)|^2 \quad . \end{aligned} \tag{4.9}$$

The ψ_n obey

$$\left\{-\omega_n{}^2(\phi_c) - \frac{d^2}{dx^2} + g\left[3\phi_c{}^2 - c^2 + \Delta\langle\phi_q{}^2\rangle\right]\right\}\psi_n = 0 \tag{4.10}$$

and the renormalized energy difference is

$$\Delta E(\text{renormalized}) = \int dx\left[\tfrac{1}{2}\dot{\phi}_c^{\,2} + \tfrac{1}{2}\left(\frac{\partial\phi_c}{\partial x}\right)^2 + \frac{g}{4}(\phi_c^{\,2}-c^2)^2\right] + \tfrac{1}{2}\sum_n\left(\omega_n(\phi_c)-\omega_n(c)\right) + \tfrac{1}{2}B\int dx\left[\phi_c(x)^2-c^2+\Delta<\phi_q^{\,2}>\right] - \frac{3g}{4}\int dx\left[\Delta<\phi_q^{\,2}>\right]^2 \quad . \tag{4.11}$$

It is easy to see that for ϕ_c obeying the boundary condition $\phi_c^{\,2}\to c^2$ exponentially as $x\to\pm\infty$, the first and the last term in (4.11) are convergent. A simple power counting indicates that the middle two terms are at most logarithmically divergent. This implies that the sum of the middle terms

$$M(\phi_c) \equiv \tfrac{1}{2}\sum_n(\omega_n(\phi_c)-\omega_n(c)) + \tfrac{1}{2}B\int dx(\phi_c^{\,2}-c^2+\Delta<\phi_q^{\,2}>) \tag{4.12}$$

can be made finite after one more subtraction around $\phi_c = c$. In Appendix B, we demonstrate that the middle term actually obeys both $M(c)=0$, and $\left.\frac{\delta M}{\delta\phi_c^{\,2}}\right|_{\phi_c=c} = 0$. Thus, it can be written as a twice-subtracted form

$$M(\phi_c) = M(\phi_c) - M(c) - \int dy\,\frac{\delta M(\phi_c)}{\delta\phi_c^{\,2}(y)}\,(\phi_c^{\,2}(y)-c^2) \quad , \tag{4.13}$$

and hence it is already finite. This implies that $\Delta E(\text{ren})$ is indeed finite. In Sec. B, we shall work out a simple example to illustrate both the renormalization and the effect of quantum fluctuation.

B. Potential Well With ϕ_c = constant

We consider a simple $\phi_c(x)$ which describes a potential as given in Fig. 1. We choose $\phi_c(x)$ such as

$$\phi_c(x) = a \ , \qquad \text{a constant} \neq c \ , \quad |x| < \frac{L}{2} - \frac{b}{2} \ , \qquad (4.14a)$$

and

$$\phi_c(x) = c \qquad |x| > \frac{L}{2} + \frac{b}{2} \ , \qquad (4.12b)$$

where $L \gg \frac{1}{\sqrt{gc^2}}$ represents the width of the well. We assume further that ϕ_c varies smoothly from $\phi_c = c$ to $\phi_c = a$ in the transition regions $\frac{L}{2} - \frac{b}{2} < |x| < \frac{L}{2} + \frac{b}{2}$, with b being a small but finite length. Under the limit of $L \to \infty$, b finite; we find as expected that the contribution due to these transition regions are negligible.

To evaluate the energy difference, one has to know how to handle the sum over states n . One can do this be rewriting the summation over n as a sum over the discrete bound state contributions plus the contribution due to the change of density of states in the continuum, as explained by Dashen et al. in Ref. 5. However, for $L \gg \frac{1}{\sqrt{gc^2}}$, we can greatly simplify the calculation by appealing to the additivity of the density of states in the configuration space as we shall explain here. The argument goes as follows: For $L \gg \frac{1}{\sqrt{gc^2}}$, the summation over states n can be written as

$$\sum_n = \sum_{|x| > \frac{L}{2} + \frac{b}{2}} + \sum_{|x| < \frac{L}{2} - \frac{b}{2}} + \sum_{\substack{\text{Transition} \\ \text{region}}}$$

$$\approx \sum_{|x| > \frac{L}{2}} + \sum_{|x| < \frac{L}{2}} \ . \qquad (4.15)$$

The contribution from the transition region is of the order $O(1)$, independent of L; and can be ignored. Also, for $|x| > \frac{L}{2}$, we have $\phi_c = c$ and the contribution due to ϕ_c and "the vacuum" will cancel each other. Thus, we have for any integrand $F(\phi_c)$

$$
\begin{aligned}
&\sum_n \Big(F(\phi_c) - F(c)\Big) \\
&\approx \sum_{n,\, |x| < \frac{L}{2}} \Big(F(a) - F(c)\Big) \\
&= \int_{\phi_c = a} L\,\frac{dk}{2\pi}\,F(a) - \int_{\phi_c = c} L\,\frac{dk}{2\pi}\,F(c) \\
&= L\left[\int_{\phi_c = a} \frac{dk}{2\pi}\,F(a) - \int_{\phi_c = c} \frac{dk}{2\pi}\,F(c)\right] \quad . \qquad (4.16)
\end{aligned}
$$

In (4.16), we have ignored terms of $O(1)$, and used the relation

$$
\sum_{n,\, |x| < \frac{L}{2}} \rightarrow \int L\,\frac{dk}{2\pi} \qquad (4.17)
$$

for both $\phi_c = a$ and $\phi_c = c$. For large L, the contribution is always proportional to L. This confirms the assertion that the phase space is additive and that the contribution from the transition region is negligible.

As an independent check of our approximation (4.15), (4.16); we have also computed the sum over states by using the method developed in Ref. 5. The direct calculation confirms our result (4.16).

Using the relation,

$$<\phi_q^{\,2}(x)>_{\phi_c} = \lim_{x'\to x} i\, G_F(x,x')\Big|_{\phi_c}$$
$$= \lim_{x'\to x} \int \frac{d^2k}{(2\pi)^2}\, e^{-ik(x'-x)} \frac{i}{k^2 - g(3\phi_c^2 - c^2 + 3\Delta<\phi_q^2>) + i\epsilon} \tag{4.18}$$

we have

$$\Delta<\phi_q^{\,2}> \equiv <\phi_q^{\,2}>\Big|_{\phi_c} - <\phi_q^{\,2}>_{\phi_c = c}$$
$$= \lim_{x'\to x}\left[i\, G_F(x,x')\Big|_{\phi_c} - i\, G_F(x,x')_{\phi_c = c}\right]$$
$$= \int \frac{d^2k}{(2\pi)^2}\left[\frac{i}{k^2 - g(3\phi_c^{\,2} - c^2 + 3\Delta<\phi_q^{\,2}>) + i\epsilon} - \frac{i}{k^2 - 2g_c^{\,2} + i\epsilon}\right]$$
$$= \frac{1}{4\pi} \ln \frac{2c^2}{3\phi_c^{\,2} - c^2 + 3\Delta<\phi_q^{\,2}>} \quad . \tag{4.19}$$

Equation (4.19) determines $\Delta<\phi_q^{\,2}>$ self-consistently. This is analogous in spirit to (2.7), (2.8) which determine $<x^2>$ self-consistently in an anharmonic oscillator. Note that $\Delta<\phi_q^{\,2}>$ determined from (4.19) is finite even though $<\phi_q^{\,2}>_{\phi_c}$ is logarithmically divergent.

For $|x| > \frac{L}{2}$, we have $\phi_c = c$, and hence

$$\Delta<\phi_q^{\,2}> = 0 \quad , \qquad |x| > \frac{L}{2} \quad . \tag{4.20}$$

For $|x| < \frac{L}{2}$, we have $\phi_c = a$, and $\Delta<\phi_q^{\,2}>$ is a constant specified by (4.19). The numerical solutions for $\Delta<\phi_q^{\,2}>$ as a function of $\phi_c^{\,2}$ and c^2 can be obtained easily.

Knowing $\Delta<\phi_q^{\,2}>$, we can compute the renormalized ground state ΔE as

$$\begin{aligned}\Delta E(\text{ren}) &= L\frac{g}{4}(a^2-c^2)^2 + \tfrac{1}{2}\sum_n \left(\omega_n(a)-\omega_n(c)\right) \\ &\quad + \tfrac{1}{2} BL(a^2-c^2+\Delta<\phi_q^{\,2}>) \\ &\quad - \frac{3g}{4} L(\Delta<\phi_q^{\,2}>)^2 \\ &= L\Big[\frac{g}{4}(a^2-c^2)^2 + \tfrac{1}{2}\int\frac{dk}{2\pi}\left(\omega_n(a)-\omega_n(c)\right) \\ &\quad + \tfrac{1}{2}B(a^2-c^2+\Delta<\phi_q^{\,2}>) - \frac{3g}{4}(\Delta<\phi_q^{\,2}>)^2\Big]. \qquad (4.21)\end{aligned}$$

The (divergent) renormalization counter term B is given by

$$\begin{aligned}B &= -3g<\phi_q^{\,2}(c)> \\ &= -3g \lim_{x'\to x}\int\frac{d^2k}{(2\pi)^2}\, e^{-ik(x'-x)}\frac{i}{k^2-2gc^2+i\epsilon} \\ &= -3g\int\frac{dk}{2\pi}\frac{1}{2(k^2+2gc^2)^{\frac{1}{2}}} \quad , \qquad (4.22)\end{aligned}$$

and the energy eigenvalues for $\phi_c=a$ and $\phi_c=c$ are

$$\omega_n(a) = \left[k^2+g(3\Delta<\phi_q^{\,2}>+3\phi_c^{\,2}-c^2)\right]^{\frac{1}{2}} \qquad (4.23a)$$

and

$$\omega_n(c) = \left[k^2+2gc^2\right]^{\frac{1}{2}} \qquad (4.23b)$$

respectively. Substituting (4.22), (4.23) into (4.21) and dividing it by L, we obtain the effective potential $V(a)$ as the energy density associated with the potential well $\phi_c=a$:

$$\begin{aligned}V(a) &= \frac{\Delta E(\text{ren})}{L} \\ &= \frac{g}{4}(a^2-c^2)^2 + \tfrac{1}{2}\int\frac{dk}{2\pi}\Big\{\left[k^2+g(3\Delta<\phi_q^{\,2}>+3a^2-c^2)\right]^{\frac{1}{2}}\end{aligned}$$

(cont.)

$$- (k^2+2gc^2)^{\frac{1}{2}} - \frac{3}{2}g(\Delta\langle\phi_q^{\,2}\rangle + a^2 - c^2)(k^2+2gc^2)^{-\frac{1}{2}}\Bigg\}$$

$$- \frac{3g}{4}(\Delta\langle\phi_q^{\,2}\rangle)^2 \quad . \tag{4.24}$$

It is now easy to see that (4.24) is finite. The expression in the curl brackets is of the form

$$f(\Delta\langle\phi_q^{\,2}\rangle + a^2) - f(c^2) - f'(c^2)(\Delta\langle\phi_q^{\,2}\rangle + a^2 - c^2) \tag{4.25}$$

which represents the removal of the first two divergent terms of f in the Taylor expansion. It is important to note that our calculation leads automatically to a Taylor expansion around $\Delta\langle\phi_q^{\,2}\rangle + \phi_c^{\,2} = c^2$ rather than around $\phi_c^{\,2} = c^2$.

Now, we wish to point out the connection of our calculation to the usual one-loop calculation of the effective potential. If we ignore the effect of quantum fluctuation by setting

$$\Delta\langle\phi_q^{\,2}\rangle \equiv \langle\phi_q^{\,2}\rangle\Big|_{\phi_c} - \langle\phi_q^{\,2}\rangle\Big|_c \to 0$$

$$\text{(no quantum fluctuation)} \quad , \tag{4.26}$$

we are led to the one-loop effective potential as

$$V_{1\text{-loop}}(a) = \frac{\Delta E(a)}{L}\Bigg|_{\Delta\langle\phi_q^{\,2}\rangle\to 0}$$

$$= \frac{g}{4}(a^2-c^2)^2 + \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\Bigg\{\Big[k^2+g(3a^2-c^2)\Big]^{\frac{1}{2}}$$

$$-(k^2+2gc^2)^{\frac{1}{2}} - \frac{3}{2}g(a^2-c^2)(k^2+2gc^2)^{-\frac{1}{2}}\Bigg\}$$

$$= \frac{g}{4}(a^2-c^2)^2 + \frac{g}{8\pi}\Big[3(a^2-c^2) - (3a^2-c^2)\ln\frac{3a^2-c^2}{2c^2}\Big] \quad . \tag{4.27}$$

When we keep $\Delta\langle\phi_q^2\rangle$ terms in (4.19) and (4.24), we go beyond the one-loop level. In terms of graphs, our calculation includes all the cactus-type diagrams as given in Fig. 2, and corresponds to a partial sum of n-loop diagrams for all n.[14] Since we obtain our results self-consistently, our model contains features which can not be achieved from calculations based on a finite order of loops. For instance, $V_{1\text{-loop}}(a)$ in (4.27) is complex for small a. This unphysical result reflects the inadequacy of the one-loop calculation. Since loop-wise summation can be viewed as a perturbation by orders in $\hbar$, it is easy to see that including contributions up to any finite order will not make the resultant amplitude real. On the other hand, $\Delta\langle\phi_q^2\rangle$ and V(a) determined in (4.19) and (4.24) in our model are finite and real.

In the following, we wish to present our numerical solutions for $\Delta\langle\phi_q^2\rangle$ and V(a) as functions of a and the renormalization point c. In Fig. 3a, we plot the numerical solution of $\Delta\langle\phi_q^2\rangle$. With $\Delta\langle\phi_q^2\rangle$ given, V(a) can be evaluated explicitly from (4.24) as

$$V(a) = \frac{g}{4}(a^2-c^2)^2 + \frac{gc^2}{2\pi}\left[\frac{3(\Delta\langle\phi_q^2\rangle + a^2 - c^2)}{4c^2} - \frac{3a^2+3\Delta\langle\phi_q^2\rangle - c^2}{4c^2}\ln\frac{3a^2+3\Delta\langle\phi_q^2\rangle - c^2}{2c^2}\right] - \frac{3g}{4}(\Delta\langle\phi_q^2\rangle)^2 \quad . \tag{4.28}$$

The numerical solution to V(a) is given in Fig. 3b,c.

Several features of these solutions are worth mentioning:

(1) $\phi_c=0$ is always a local minimum of $V(\phi_c)$;

(2) $\phi_c=c$ is stationary point of $V(\phi_c)$.

It is a local minimum for $4\pi c^2 > 3$, and becomes a local maximum for $4\pi c^2 < 3$. At $4\pi c^2=3$, it becomes a point of inflection;

(3) The minimum at $\phi_c=c$ represents the true ground state for $4\pi c^2 > 5.1332$, and $\phi_c=0$ is the true ground state for $5.1332 > 4\pi c^2 > 3$. Since, for $4\pi c^2 < 3$, the stationary point at $\phi_c=c$ is a local maximum, it does not make any sense to do perturbation around this point. However, a new minimum appears at a different location for $\phi_c > c$. In this case, we shall compare the new minimum with the minimum at $\phi_c=0$. The position of the new minimum can be determined by the zero of

$$\frac{\partial V(a)}{\partial a} = g(a^2-c^2)a + 3ga\Delta\langle\phi_q^2\rangle \quad . \tag{4.29}$$

The physical meaning of the above results based on the parameter c is not immediately transparent. We wish to translate the conclusion in terms of the physical coupling constant g and the mass. In two-dimensional ϕ^4 theory, the coupling constant g has a dimension of $(\text{mass})^2$. Thus, the intrinsic strength of the coupling should be described by $g/(\text{mass})^2$. Since the effective potential in our model has two distinct minima, it provides two natural mass scales. The intrinsic coupling strength measured by these two different ground state mass scales are usually different.

Consider the effective potential around the abnormal ground state at $\phi_c = c$. By expanding the solution around $\phi_c = c$, we have

$$\Delta \langle \phi_q^2 \rangle = -\frac{6c(\phi_c - c)}{8\pi c^2 + 3} + O\left((\phi_c - c)^2\right), \tag{4.30}$$

and

$$V(\phi_c) = gc^2 \left(1 - \frac{9}{8\pi c^2 + 3}\right) (\phi_c - c)^2 + O\left((\phi_c - c)^3\right) . \tag{4.31}$$

Introducing a mass parameter around $\phi_c = c$ by

$$V(\phi_c) = \tfrac{1}{2} m_c^2 (\phi_c - c)^2 + O\left((\phi_c - c)^3\right) , \tag{4.32}$$

we have

$$\begin{aligned} m_c^2 &= 2gc^2 \left(1 - \frac{9}{8\pi c^2 + 3}\right) \\ &= 2gc^2 \cdot \frac{8\pi c^2 - 6}{8\pi c^2 + 3} . \end{aligned} \tag{4.33}$$

The intrinsic strength measured in terms of m_c^2 is,

$$g_c \equiv \frac{g}{m_c^2} = \frac{8\pi c^2 + 3}{4c^2 (4\pi c^2 - 3)} . \tag{4.34}$$

We find that a large c corresponds to a weak g_c. As $4\pi c^2$ decreases and approaches 3, g_c increases and approaches infinity. Thus, in terms of g_c, the ground state associated with the classical solution $\phi_c = c$ represents the true ground state in the weak coupling limit. As the intrinsic coupling g_c becomes stronger, the normal ground state at $\phi_c = 0$ can have a lower energy and becomes the true ground state. The transition occurs at $g_c = 11.957$.

It is interesting also to investigate the physical picture based on the intrinsic strength measured by the mass defined around $\phi_c=0$. Near $\phi_c=0$, we introduce a mass through

$$V(\phi_c) = \tfrac{1}{2} m_0^2 \phi_c^2 + \cdots \tag{4.35}$$

and obtain, with the help of (4.29),

$$\begin{aligned} m_0^2 &\equiv \left.\frac{d^2V(a)}{da^2}\right|_{a=0} \\ &= g\left[3\Delta\left.\langle\phi_q^2\rangle\right|_{a=0} - c^2\right] . \end{aligned} \tag{4.36}$$

The intrinsic strength measured in m_0^2 is

$$g_0 \equiv \frac{g}{m_0^2} = \frac{1}{3\Delta\langle\phi_q^2\rangle_0 - c^2} \tag{4.37}$$

which can be evaluated numerically as a function of c . According to the numerical calculation, we find that a large c leads to a large g_0 . However, as $4\pi c^2$ decreases and approaches 3, g_0 also decreases and approaches a limit value 9.045. A further decrease of $4\pi c^2$ leads to an increasing g_0 . The weak coupling case of $g_0 < 9.045$ can not be obtained for any c . This appears to be a rather surprising result. To understand this point, we make a similar calculation of the effective potential based on an expansion around $\phi_c=0$. We find that for $g_0 < 9.045$, the effective potential has only one minimum located at $\phi_c=0$. Now, it is obvious why $g_0 < 9.045$ can never be reached from the expansion around a minimum at $\phi_c \neq 0$. For $g_0 > 9.045$, a second minimum appears and its position is always at $\phi_c^2 \geq 3/4\pi$. The second minimum becomes the true ground state when $g_0 \geq 10.211$. The effective potential as a function of g_0 is shown in Fig. 4.

In terms of g_0, it is interesting to see that the ground state at $\phi_c = 0$ is stable only if the coupling is relatively weak (i.e., only if $g_0 \leq 10.211$). It too becomes unstable if the coupling becomes strong. There is of course no contradiction between these two descriptions. It follows from the fact that a large g_c corresponds to a small g_0 and vice versa.[15] It also suggests that a boson ground state will always become unstable if the intrinsic coupling associated with this ground state becomes too strong. Then, it will jump to an alternative ground state with a weaker associated intrinsic coupling strength.

V. A POSSIBLE GENERALIZATION TO ϕ^4 THEORY IN THREE SPACE AND ONE TIME DIMENSION

We now consider the ϕ^4 Lagrange function in three space and one time dimension,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{4}(\phi^2 - c^2)^2 \quad .$$

The field equation, the Hamiltonian, and the quantization rules associated with this theory are all given in Sec. III. It is well-known that for a ϕ^4 theory in 4 dimensions, we need to make, in addition to the mass renormalization, the coupling constant and the wave function renormalizations as well. However, in our approximation developed in the previous sections, we include only the cactus-type diagrams in computing the effective potential. These diagrams do not lead to any nontrivial wave function and coupling constant renormalizations. Hence, further modifications are needed in order to arrive at a finite result.[16] Using the one dimensional theory as a guide, we propose the following generalization of our model to the three dimensional theory:

1. We assume that the equations for ϕ_c and ϕ_q are essentially unmodified: (See (4.7) and (4.8))

$$\partial^2\phi_c + g(\phi_c^{\ 2} - c^2)\phi_c + 3g\,\Delta\langle\phi_q^{\ 2}\rangle\phi_c = 0 \quad , \tag{5.2}$$

$$\partial^2\phi_q + g(3\phi_c^{\ 2} - c^2)\phi_q + 3g\,\Delta\langle\phi_q^{\ 2}\rangle\phi_q = 0 \quad . \tag{5.3}$$

Equations (5.2) and (5.3) are equivalent to the approximations

$$\phi_q^{\ 2} + \text{counter terms} \rightarrow \Delta\langle\phi_q^{\ 2}\rangle \quad , \tag{5.4}$$

and

$$\phi_q^{\ 3} + \text{counter terms} \rightarrow 3\Delta\langle\phi_q^{\ 2}\rangle\phi_q \quad . \tag{5.5}$$

2. To obtain a finite $\Delta\langle\phi_q^{\ 2}\rangle$, we assume that $\Delta\langle\phi_q^{\ 2}\rangle$ is given by a doubly subtracted expression:

$$\begin{aligned}\Delta\langle\phi_q^{\ 2}\rangle &\equiv \langle\phi_q^{\ 2}\rangle - \langle\phi_q^{\ 2}\rangle_{U=0} - \int dy \left.\frac{\delta\langle\phi_q^{\ 2}\rangle}{\delta U(y)}\right|_{U=0} U(y) \\ &= \lim_{x'\to x}\Bigg[i\,G_F(x,x') - i\,G_F(x,x')\Big|_{U=0} \\ &\qquad - i\int dy \left.\frac{\delta G_F(x,x')}{\delta U(y)}\right|_{U=0} U(y)\Bigg]\end{aligned} \tag{5.6}$$

where G_F is the Green's function associated with (5.3) and obeys

$$\left[\partial^2+2gc^2+U(x)\right]G_F(x,x') = -\delta^4(x-x') \qquad (5.7)$$

with

$$U(x) = 3g(\phi_c^{\,2}+\Delta<\phi_q^{\,2}>-c^2) \qquad (5.8)$$

G_F also obeys a similar equation for x'.

3. The ground state energy difference associated with an arbitrary ϕ_c (with $\dot{\phi}_c = 0$) is,

$$\begin{aligned}\Delta E(\phi_c) = &\int dx\left[\tfrac{1}{2}(\nabla\phi_c)^2 + \tfrac{g}{4}(\phi_c^{\,2}-c^2)^2\right]\\ &+\tfrac{1}{2}\sum_n\left[\omega_n(U)-\omega_n(U=0) - \int dy \left.\frac{\delta\,\omega_n}{\delta\,U(y)}\right|_{U=0} U(y)\right.\\ &\left.-\int dy\,dz\left.\frac{\delta^2\omega_n}{\delta\,U(y)\delta\,U(z)}\right|_{U=0} U(y)U(z)\right]\\ &-\frac{3g}{4}\int dx(\Delta<\phi_q^{\,2}>)^2 \quad . \end{aligned} \qquad (5.9)$$

Additional subtractions are made in (5.6) and (5.9) in order to obtain finite results. These additional subtractions are put in by hand, and do not correspond to simple counter terms in the Lagrange function. For this reason, the conclusion arrived at in this section is less reliable than the one dimensional result. Note also that the subtractions in (5.6) and (5.9) are carried out in the Taylor's expansion of the external potential $U(x)\left(=3g(\phi_c^{\,2}+\Delta<\phi_q^{\,2}>-c^2)\right)$, rather than of the classical field $\phi_c(x)-c$.

Once we accept this subtraction scheme, we can compute the effective potential $V(a)$, and consequently, study the stability of the ground state. To test the consistency of our subtraction scheme, we compare our result with the loop expansion. Just as in the one dimensional theory, if we suppress the quantum fluctuation term $\Delta\langle\phi_q^2\rangle$ in $V(a)$, we reproduce the one loop calculation as given in Ref. 13. This indicates that our subtraction scheme is at least consistent with the standard method at the one loop level. When the quantum fluctuations are included, the effective potential becomes

$$
\begin{aligned}
V(a) &\equiv \frac{\Delta E}{U} \\
&= \frac{g}{4}(a^2-c^2)^2 + \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left\{\left[k^2+2gc^2+3g(a^2+3\langle\phi_q^2\rangle-c^2)\right]^{\frac{1}{2}}\right. \\
&\qquad \left. -(k^2+2gc^2)^{\frac{1}{2}} - \text{subtraction terms}\right\} \\
&\qquad -\frac{3g}{4}(\Delta\langle\phi_q^2\rangle)^2 \\
&= \frac{g}{4}(a^2-c^2)^2 - \frac{3g}{4}(\Delta\langle\phi_q^2\rangle)^2 + \frac{g^2}{64\pi^2}\left[(3a^2+3\Delta\langle\phi_q^2\rangle - c^2)\right. \\
&\qquad \left.\times \ln\frac{3a^2+3\Delta\langle\phi_q^2\rangle-c^2}{2c^2} - \frac{3}{2}(a^2+\Delta\langle\phi_q^2\rangle-c^2)(9a^2+9\Delta\langle\phi_q^2\rangle-5c^2)\right]
\end{aligned}
\tag{5.10}
$$

where the quantum fluctuation term is determined self-consistently by

$$
\begin{aligned}
\Delta\langle\phi_q^2\rangle &= \int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2-2gc^2-3g(a^2+\Delta\langle\phi_q^2\rangle-c^2)+i\epsilon}\right. \\
&\qquad \left. -\frac{i}{k^2-2gc^2+i\epsilon} - \text{subtraction term}\right]
\end{aligned}
$$

(cont.)

$$= \frac{g}{16\pi^2}\left[(3a^2+3\Delta\langle\phi_q{}^2\rangle - c^2)\ln\frac{3a^2+3\Delta\langle\phi_q{}^2\rangle - c^2}{2c^2} + 3(c^2-a^2-\Delta\langle\phi_q{}^2\rangle)\right] \quad . \tag{5.11}$$

we can evaluate $V(a)$ and $\Delta\langle\phi_q{}^2\rangle$ as functions of g and c^2 by solving numerically (5.10), (5.11). The results are plotted in Fig. 5. These solutions share many features of the one dimensional solutions. In particular, as we increase the coupling strength g, the ground state at $\phi_c = c$ becomes unstable and a first order phase transition occurs as the coupling constant reaches a critical value given by $g = 62.385$.

It is probably worth noting that (5.11) has a self-consistent solution only in the region where $\phi_c{}^2 = a^2$ is smaller than, or of the same order as c^2. For $\phi_c{}^2$ larger than a certain critical value, (5.11) no longer possesses a real self-consistent solution, indicating that our approximation breaks down in this limit.[17] At the moment, we do not know whether the subtraction scheme proposed here for the three dimensional theory is faithful and self-consistent; nor do we know whether it obeys the multiplicative renormalization. We plan to investigate these problems in the future.

VI. INTERNAL SYMMETRY

In this section, we shall discuss briefly the framework through which internal symmetry can be introduced into our model. We take an O(n) symmetry as an example. Consider an O(n) invariant Lagrange function,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{g}{4}(\vec{\phi}^{\,2} - c^2)^2 \tag{6.1}$$

where $\vec{\phi} = (\phi^1, \phi^2, \cdots, \phi^n)$ transforms as the fundamental n-dimensional representation of the group. The field equation is

$$\partial^2 \phi^i + g(\vec{\phi}^{\,2} - c^2)\phi^i = 0 \quad . \tag{6.2}$$

Separating the field operator ϕ^i into a c-number and an operator part,

$$\phi^i = \phi_c{}^i + \phi_q{}^i \tag{6.3}$$

and making the approximation,

$$\phi_q{}^i \phi_q{}^j \rightarrow \langle \phi_q{}^i \phi_q{}^j \rangle \quad , \tag{6.4}$$

$$\begin{aligned} \phi_q{}^i \phi_q{}^j \phi_q{}^k \rightarrow & \langle \phi_q{}^i \phi_q{}^j \rangle \phi_q{}^k + \langle \phi_q{}^j \phi_q{}^k \rangle \phi_q{}^i \\ & + \langle \phi_q{}^k \phi_q{}^i \rangle \phi_q{}^j \quad ; \end{aligned} \tag{6.5}$$

we obtain

$$\partial^2 \phi_c{}^i + g(\vec{\phi}_c{}^2 - c^2)\phi_c{}^i + g\left[\langle \vec{\phi}_q{}^2 \rangle \delta^{ij} + 2\langle \phi_q{}^i \phi_q{}^j \rangle\right]\phi_c{}^j = 0 \tag{6.6}$$

and

$$\begin{aligned} \partial^2 \phi_q{}^i & + g(\vec{\phi}_c{}^2 \delta^{ij} + 2\phi_c{}^i \phi_c{}^j - c^2 \delta^{ij})\phi_q{}^j \\ & + g\left[\langle \vec{\phi}_q{}^2 \rangle \delta^{ij} + 2\langle \phi_q{}^i \phi_q{}^j \rangle\right]\phi_q{}^j = 0 \quad . \end{aligned} \tag{6.7}$$

Equations (6.7) and (6.8) are direct generalizations of (3.11) and (3.12). Since (6.8) is linear in the field operator $\phi_q{}^i$, we can introduce the creation and annihilation operators as before

$$\phi_q{}^i(x) = \sum_n \left(\psi_n^{i*}(x) e^{i\omega_n t} a_n{}^\dagger + \psi_n{}^i(x) e^{-i\omega_n t} a_n \right) \tag{6.8}$$

where the c-number (multi-component) wave function $\{\psi_n{}^i\}$ obeys

$$\Big\{(-\omega_n{}^2-\nabla^2)\delta^{ij}+g(\vec{\phi}_c{}^2\delta^{ij}+2\phi_c{}^i\phi_c{}^j-c^2\delta^{ij}) + g\Big[\langle\vec{\phi}_q{}^2\rangle\delta^{ij}+2\langle\phi_q{}^i\phi_q{}^j\rangle\Big]\Big\}\psi_n{}^j = 0 \qquad (6.9)$$

with the normalization condition

$$2\omega_n\int dx\, \vec{\psi}_n^{\,*}\cdot\vec{\psi}_m = \delta_{nm} \quad . \qquad (6.10)$$

The analog of the Hamiltonian (3.24) is

$$H = H_c + H_{cq} + H_q \qquad (6.11)$$

with

$$H_c = H(\phi_c)_{\text{classical}} \quad , \qquad (6.12)$$

$$H_{cq} = \int dx(\dot{\phi}_c{}^i\dot{\phi}_q{}^i - \ddot{\phi}_c{}^i\phi_q{}^i) \qquad (6.13)$$

and

$$H_q = \sum_n (a_n{}^\dagger a_n + \tfrac{1}{2})\omega_n - \frac{g}{4}\int dx\Big[\langle\vec{\phi}_q{}^2\rangle^2 + 2\langle\phi_q{}^i\phi_q{}^j\rangle^2\Big] \quad . \qquad (6.14)$$

For the ground state, the vacuum expectation values $\langle\phi_q{}^i(x)\phi_q{}^j(x)\rangle$ can be determined through

$$\langle\phi_q{}^i(x)\phi_q{}^j(x)\rangle = \lim_{x'\to x} T\langle\phi_q{}^i(x)\phi_q{}^j(x')\rangle = \lim_{x'\to x} G_F^{ij}(x,x') \qquad (6.15)$$

where $G_F{}^{ij}$ obeys

$$\Big\{\delta^{ij}\partial^2 + g\Big[\vec{\phi}_c^{\,2}\delta^{ij} + 2\phi_c^{\,i}\phi_c^{\,j} - c^2\delta^{ij}\Big] + g\Big[<\vec{\phi}_q^{\,2}>\delta^{ij} + 2<\phi_q^{\,i}\phi_q^{\,j}>\Big]\Big\} G_F^{\,jk}(x, x') = -\delta^{ij}\delta(x - x') \quad . \tag{6.16}$$

Thus, there is no conceptual difficulty in computing $<\phi_q^{\,i}\phi_q^{\,j}>$, the ground state energy difference, the effective potential, $\cdots$. However, in practice, the introduction of internal symmetry will make the calculation far more involved.

The program of renormalization can be introduced as before. Since it is a straightforward generalization of results presented in Sec. IV, we shall not reproduce it here. We leave it as an exercise for serious readers.

VII. DISCUSSION

In this paper, we have studied the effect of quantum fluctuations on the stability of the vacuum in a self-interacting boson theory. In both the one and the three dimensional theory, the abnormal vacuum state becomes unstable as the coupling becomes stronger. A first order transition from the abnormal to the normal vacuum occurs as the coupling reaches a critical value. One natural extension of the present work is to study the stability of the vacuum in the presence of fermions. It is known in certain systems with nonzero fermion density that the contribution of the fermions to the ground state energy is opposite to that of the bosons.[18] One might expect that the same effect may also appear in the vacuum energy, and hence the presence of fermions should tend to stabilize the abnormal vacuum. Yan and the author have investigated this problem, and found that this is indeed the case. The details of the calculation will appear in a separate publication.

We conclude this paper by listing a few important questions which remain to be answered:

1. Can we develop a systematic method of improving the Hartree approximation by including more and more higher order quantum fluctuation terms? The method should preserve the variational principle, and leads to, in principle, an exact method of evaluating the energies and wave functions of various systems.

2. How do we formulate the multiparticle solutions, and describe the scattering phenomena in our framework? Since in our model the one-particle state lives in a self-generated bag, a description of the multiparticle states should include the interaction among the bags.

3. To make contact with the real world, we have to include fermions, introduce various internal symmetries, and preserve PCAC. It is well-known that the inclusion of PCAC imposes a serious challenge to the existing bag models. Does our model provide a clue to this important question?[19]

ACKNOWLEDGMENTS

The author wishes to thank his colleagues at the University of Illinois and the Fermilab for many helpful discussions. He wishes to acknowledge with pleasure the collaboration of J. A. Wright on the one mode approximation, which initiated the present investigation. The author is indebted to B. Lee for suggesting him to test the approximation method on the quantum anharmonic oscillator, to S. D. Ellis for a critical reading of the manuscript, and to R. Dashen and T. M. Yan for stimulating discussions. Finally, he wishes to thank B. Lee and the members of the Theory Group at Fermilab for their hospitality.

APPENDIX A: VARIATIONAL PRINCIPLE

In this appendix, we show that Eq. (3.11) can be obtained from a variational principle. To establish the connection, it is convenient to rewrite the energy (3.31) (See also (3.25)-(3.28)) as

$$E = \sum \frac{C_n \omega_n}{2} + \int dx \Big[\tfrac{1}{2} (\nabla \phi_c)^2 + \frac{g}{4} (\phi_c^2 - c^2)^2$$
$$+ \tfrac{1}{2} \sum_n C_n \psi_n^* (-\omega_n^2 - \nabla^2) \psi_n$$
$$+ \tfrac{1}{2} g(3\phi_c^2 - c^2) \sum_n C_n |\psi_n|^2 + \frac{3g}{4} \Big(\sum_n C_n |\psi_n|^2\Big)^2 \Big] \qquad (A.1)$$

with

$$C_n = 2N_n + 1 \quad . \qquad (A.2)$$

Equation (A.1) reduces to (3.31) trivially with the help of (3.14). Now, we have

$$\frac{\delta E}{\delta \phi_c(x)} = \left(\frac{\delta E}{\delta \phi_c(x)}\right)_{\text{explicit}} + \sum_n \frac{\delta E}{\delta \omega_n} \frac{\delta \omega_n}{\delta \phi_c(x)}$$
$$+ \sum_n \int dy \frac{\delta E}{\delta \psi_n(y)} \frac{\delta \psi_n(y)}{\delta \phi_c(x)} + \sum_n \int dy \frac{\delta E}{\delta \psi_n^*(y)} \frac{\delta \psi_n^*(y)}{\delta \phi_c(x)} \quad . \qquad (A.3)$$

Using (3.14), (3.17) and (3.21), we see that

$$\frac{\delta E}{\delta \omega_n} = \frac{C_n}{2} \Big(1 - 2\omega_n \int dx |\psi_n|^2\Big) = 0 \qquad (A.4)$$

$$\frac{\delta E}{\delta \psi_n^*(y)} = \frac{C_n}{2} \Big[(-\omega_n^2 - \nabla^2)\psi_n + g(3\phi_c^2 - c^2)\psi_n$$
$$+ 3g\Big(\sum_m C_m |\psi_m|^2\Big)\psi_n \Big] = 0 \quad , \qquad (A.5)$$

and similarly

$$\frac{\delta E}{\delta \psi_n(y)} = 0 \quad . \tag{A.6}$$

Hence,

$$\begin{aligned} \frac{\delta E}{\delta \phi_c(x)} &= \left(\frac{\delta E}{\delta \phi_c(x)}\right)_{\text{explicit}} \\ &= -\nabla^2 \phi_c + g(\phi_c{}^2 - c^2)\phi_c + 3g \sum_n C_n |\psi_n|^2 \phi_c \\ &= -\nabla^2 \phi_c + g(\phi_c{}^2 - c^2)\phi_c + 3g \langle \phi_q{}^2 \rangle \phi_c \quad . \end{aligned} \tag{A.7}$$

Thus, the variational result

$$\frac{\delta E}{\delta \phi_c(x)} = 0 \tag{A.8}$$

leads precisely to (3.11) as desired.

The variational principle can be generalized in a straightforward fashion to include the effects of renormalization. The only important modification is to replace the energy E by the renormalized energy.

APPENDIX B: RENORMALIZABILITY IN ONE SPACE AND ONE TIME DIMENSION

In Sec. IV, we indicate that the energy density $\Delta E(\phi_c)$ ($\phi_c(x) \to c$ exponentially as $x \to \pm\infty$) is finite if

$$M(\phi_c^2) \equiv \sum_n \omega_n(\phi_c) - \sum_n \omega_n(c) + B\int dy\left[\phi_c(y)^2 - c^2 + \sum_n |\psi_n(\phi_c)|^2 - \sum_n |\psi_n(c)|^2\right] \tag{B.1}$$

is finite. We also show that M is finite after at most two subtractions. In this Appendix, we wish to show that both $M(\phi_c^2)$ and $\dfrac{\delta M(\phi_c^2)}{\delta \phi_c^2}$ vanish at $\phi_c^2 = c^2$. Thus, M is already twice subtracted, and hence is finite.

To proceed, we start with Eq. (4.10)

$$\left[-\omega_n^2 - \frac{d^2}{dx^2} + 2gc^2 + \Delta U(\phi_c, x)\right]\psi_n(x) = 0 \tag{B.2}$$

where

$$\Delta U(\phi_c, x) \equiv 3g\left[(\phi_c^2(x) - c^2) + \sum_n |\psi_n(\phi_c, x)|^2 - \sum_n |\psi_n(c, x)|^2\right]. \tag{B.3}$$

Multiplying (B.2) by ψ_n^* and integrated over x, we obtain

$$-\omega_n^2\int dx|\psi_n|^2 + \int dx\left(\left|\frac{d\psi_n}{dx}\right|^2 + (2gc^2 + \Delta U)|\psi_n|^2\right) = 0 \tag{B.4}$$

or

$$\omega_n^2 = \frac{\int dx\left(\left|\frac{d\psi_n}{dx}\right|^2 + (2gc^2+\Delta U)|\psi_n|^2\right)}{\int dx|\psi_n|^2} \quad . \tag{B.5}$$

By differentiation, we find

$$\frac{\delta \omega_n^2}{\delta \psi_n^*(y)} = \frac{-\frac{d^2\psi_n}{dy^2} + (2gc^2+\Delta U)\psi_n(y)}{\int dx |\psi_n|^2}$$

$$- \frac{\int dx\left(\left|\frac{d\psi_n}{dx}\right|^2 + (2gc^2+\Delta U)|\psi_n|^2\right)}{\left(\int dx |\psi_n|^2\right)^2}$$

$$= \left[\int dx |\psi_n|^2\right]^{-1}\left[-\frac{d^2\psi_n}{dy^2} + (2gc^2+\Delta U)\psi_n(y) + \omega_n^2 \psi_n(y)\right]$$

$$= 0 \quad ; \qquad \text{(B. 6)}$$

and similarly

$$\frac{\delta \omega_n^2}{\delta \psi_n(y)} = 0 \quad . \qquad \text{(B. 7)}$$

By chain differentiations and with the help of (B. 6), (B. 7); we have

$$2\omega_n \frac{\delta \omega_n}{\delta \phi_c^2(x)} = \int dy \left[\frac{\delta \omega_n^2}{\delta \psi_n(y)} \frac{\delta \psi_n(y)}{\delta \phi_c^2(x)} + \frac{\delta \omega_n^2}{\delta \psi_n^*(y)} \frac{\delta \psi_n^*(y)}{\delta \phi_c^2(x)} + \frac{\delta \omega_n^2}{\delta \Delta U} \frac{\delta \Delta U(\phi_c, y)}{\delta \phi_c^2(x)}\right]$$

$$= \int dy \frac{\delta \omega_n^2}{\delta \Delta U} \frac{\delta \Delta U}{\delta \phi_c^2(x)} \quad . \qquad \text{(B. 8)}$$

Then, (B. 5) and (B. 8) imply that

$$\frac{\delta \omega_n}{\delta \phi_c^2(x)} = \frac{\int dy |\psi_n(y)|^2 \frac{\delta \Delta U}{\delta \phi_c^2(x)}}{2\omega_n \int dx |\psi_n|^2}$$

(cont.)

$$= \int dy \, |\psi_n(y)|^2 \frac{\delta \Delta U}{\delta \phi_c^{\,2}(x)} \quad . \tag{B.9}$$

Summing (B.9) over n, and noting that $\sum_n |\psi_n(\phi_c = c, y)|^2$ is a constant independent of y, we have

$$\left.\frac{\delta \sum_n \omega_n}{\delta \phi_c^{\,2}(x^2)}\right|_{\phi_c^{\,2}=c^2} = \int dy \sum_n |\psi_n(c)|^2 \left.\frac{\delta \Delta U}{\delta \phi_c^{\,2}(x)}\right|_{\phi_c^{\,2}=c^2}$$

$$= \sum_n |\psi_n(c)|^2 \frac{\delta}{\delta \phi_c^{\,2}(x)} \left.\int dy \, \Delta U(\phi_c, y)\right|_{\phi_c^{\,2}=c^2}$$

$$= \sum_n |\psi_n(c)|^2 3g \frac{\delta}{\delta \phi_c^{\,2}(x)} \int dy \Big[\phi_c(y)^2 - c^2$$

$$\left. + \sum_n |\psi_n(\phi_c, y)|^2 - \sum_n |\psi_n(c)|^2\Big]\right|_{\phi_c^{\,2}=c^2}$$

$$= -B \frac{\delta}{\delta \phi_c^{\,2}(x)} \int dy \Big[\phi_c(y)^2 - c^2 + \sum_n |\psi_n(\phi_c, y)|^2$$

$$\left. - \sum_n |\psi_n(c)|^2\Big]\right|_{\phi_c^{\,2}=c^2} \quad . \tag{B.10}$$

Thus, we find both

$$M(c^2) = \Big\{\sum_n \omega_n(\phi_c) - \sum_n \omega_n(c) + B\int dy \Big[\phi_c(y)^2 - c^2$$

$$+ \sum |\psi_n(\phi_c)|^2 - \sum |\psi_n(c)|^2\Big]\Big\}_{\phi_c^{\,2}=c^2} = 0 \quad , \tag{B.11}$$

and

$$\left.\frac{\delta M(\phi_c)}{\delta \phi_c^{\ 2}}\right|_{\phi_c^{\ 2}=c^2} = \left.\frac{\delta \sum_n \omega_n(\phi_c)}{\delta \phi_n^{\ 2}(x)}\right|_{\phi_c^{\ 2}=c^2}$$

$$+ B \frac{\delta}{\delta \phi_c^{\ 2}(x)} \int dy \left[\phi_c(y)^2 - c^2 + \sum_n |\psi_n(\phi_c, y)|^2 \right.$$

$$\left.\left. - \sum_n |\psi_n(c)|^2 \right]\right|_{\phi_c^{\ 2}=c^2} = 0 \quad . \qquad \text{(B.12)}$$

In other words, we can rewrite $M(\phi_c^{\ 2})$ as the twice subtracted form

$$M(\phi_c^{\ 2}) = M(\phi_c^{\ 2}) - M(c^2) - \int dy \left.\frac{\delta M(\phi_c)}{\delta \phi_c^{\ 2}(y)}\right|_{\phi_c=c} (\phi_c^{\ 2}(y) - c^2) \quad ,$$

and hence it is finite.

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[11]These two formulae are valid only if $<x> = 0$. For $<x> \neq 0$, we should have instead

$$x^4 \to 6<x^2>x^2 - 8<x>^3 x + 6<x>^4 - 3<x^2>^2 ,$$

and

$$x \to 3<x^2>x - 2<x>^3 .$$

[12]As far as the ground state energy is concerned, the approximate E_0 is the same as those obtained from a variational calculation by using a gaussian trial function. The author wishes to thank Professor T.M. Yan for pointing out this connection to him.

[13]See e.g., S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

[14]It is interesting to note that the cactus diagrams are also dominant in the O(N) model for large N . Thus, the set of equations obtained in the O(N) model are similar to ours. See, S. Coleman, R. Jackiw, and H. Politzer, Phys. Rev. D10, 2491 (1974); R.G. Root, Phys. Rev. D10, 3322 (1974) and to be published; H.J. Schnitzer, Phys. Rev. D10, 1800; 2042 (1974). The last author also studied the stability of the vacuum in the $N \to \infty$ limit.

[15]After the completion of this paper, A. Neveu informed us that this result was also known to S. Coleman. The author wishes to thank A. Neveu for transmitting this (unpublished) information to him.

[16]The O(N) model mentioned in Ref. 14 may provide an alternative method of introducing renormalizations within the Hartree approximation. In

particular, one needs to introduce bubble diagrams in the four point function and leads to a nontrivial coupling constant renormalization.

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FIGURE CAPTIONS

Fig. 1 The classical field $\phi_c(x)$ used in computing the effective potential $U(a)$.

Fig. 2 A typical cactus diagram included in our calculation.

Fig. 3 Numerical results of the fluctuation function $\Delta\langle\phi_q^2\rangle$ and the effective potential $V(a)$ in one space and one time dimension. The calculations are based on renormalizations at $\phi_c=c$.

a. $\Delta\langle\phi_q^2\rangle$ vs. a^2/c^2 for $4\pi c^2=0,1,3,$ and 6;

b. $V(a)$ vs. a/c for $4\pi c^2=3,4,5,$ and 6;

c. $V(a)$ vs. a/c for $4\pi c^2=1$. In this case, a new minimum is developed near $a/c=2.6$.

Fig. 4 Numerical results of the effective potential in one dimensional theory at the intrinsic coupling strength $g_0(=g/m_0^2)$ = 0, 5, 10, and 16 . The calculations are based on expansion at $\phi_c=0$ with the mass at $\phi_c=0$ being m .

Fig. 5 Numerical results of the effective potential in three dimensional theory with an expansion around $\phi_c=c$. The effective potential becomes complex as a/c is larger than a critical value. See Footnote 17 .

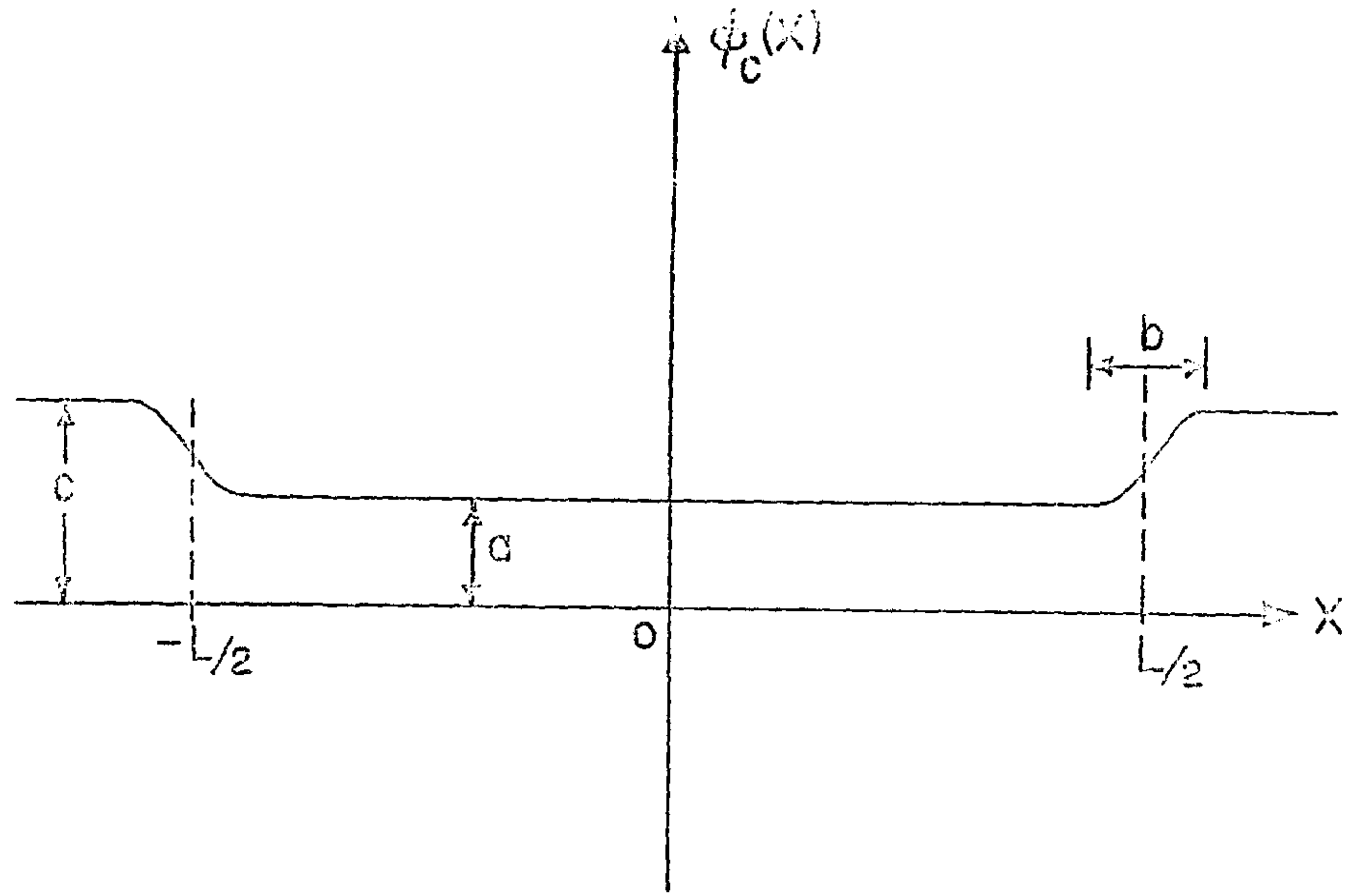


Fig. 1

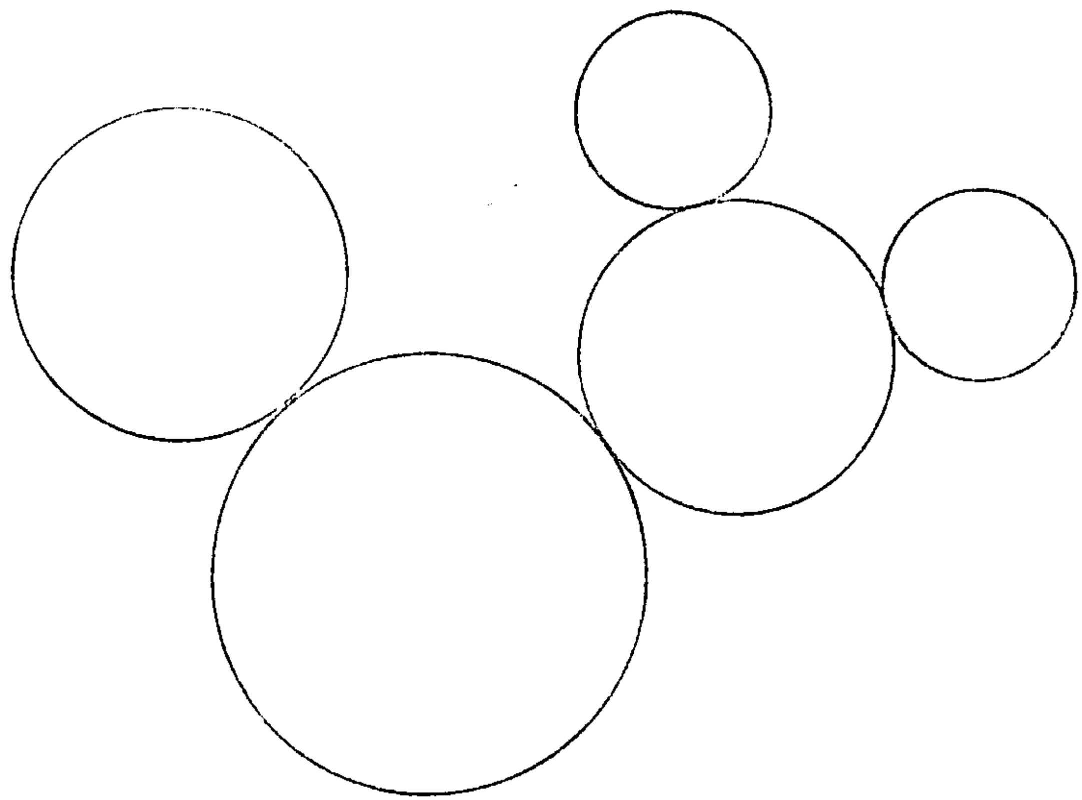
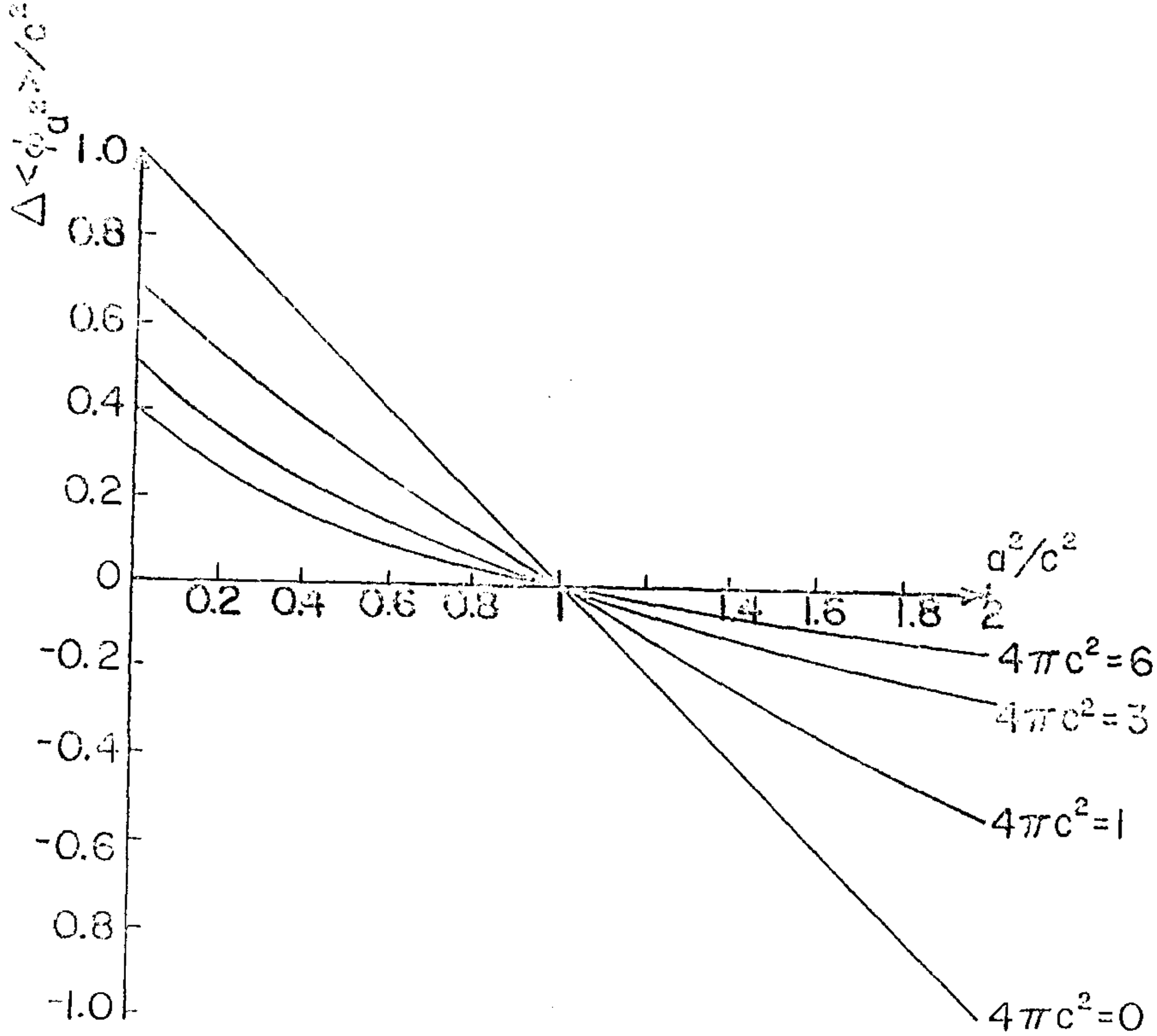


Fig. 2



1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0.8
-1.0
0.2
0.4
0.6
0.8
1
1.4
1.6
1.8
2
a^2/c^2
$4\pi c^2=6$
$4\pi c^2=3$
$4\pi c^2=1$
$4\pi c^2=0$

Fig. 3a

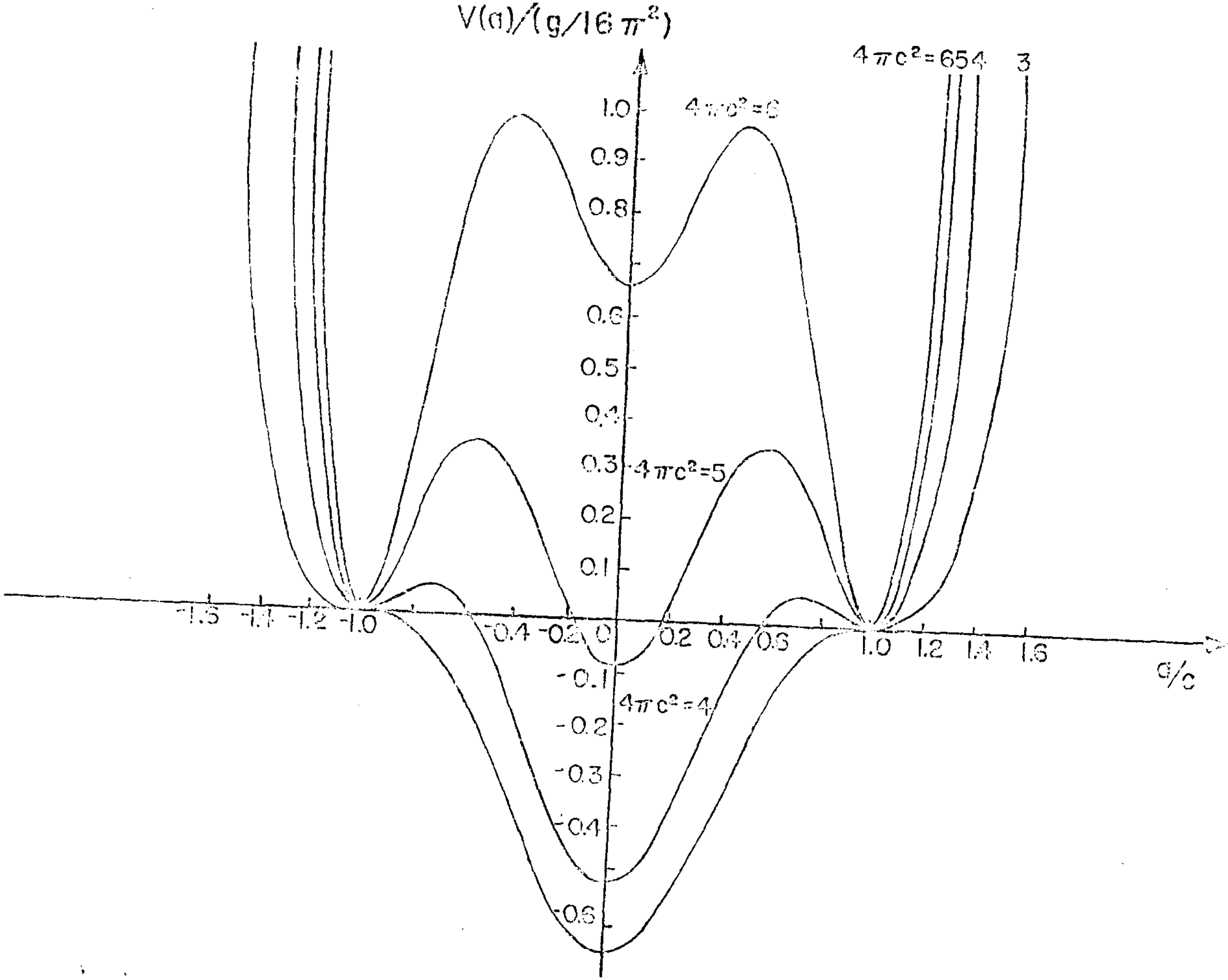


Fig. 3b

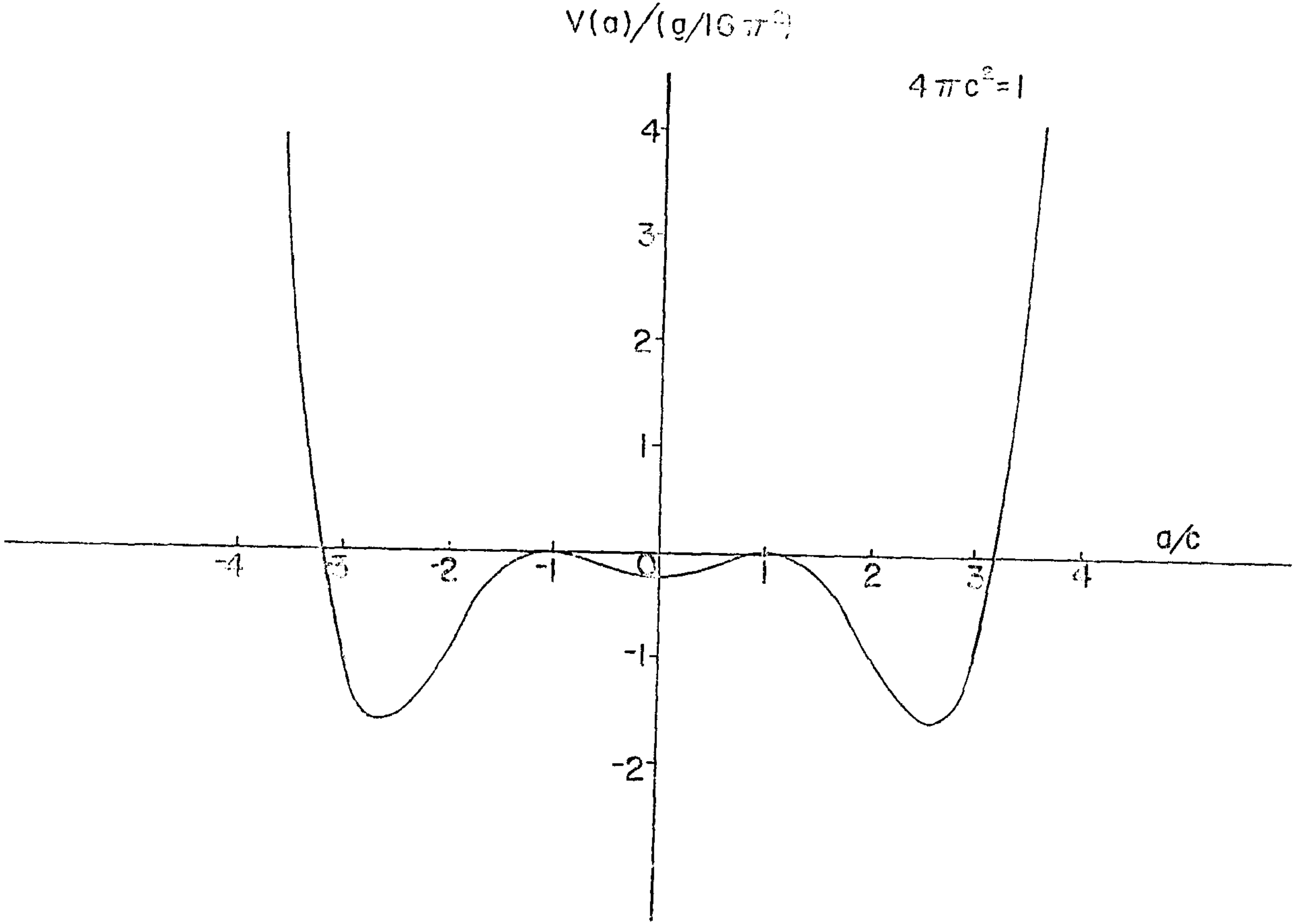
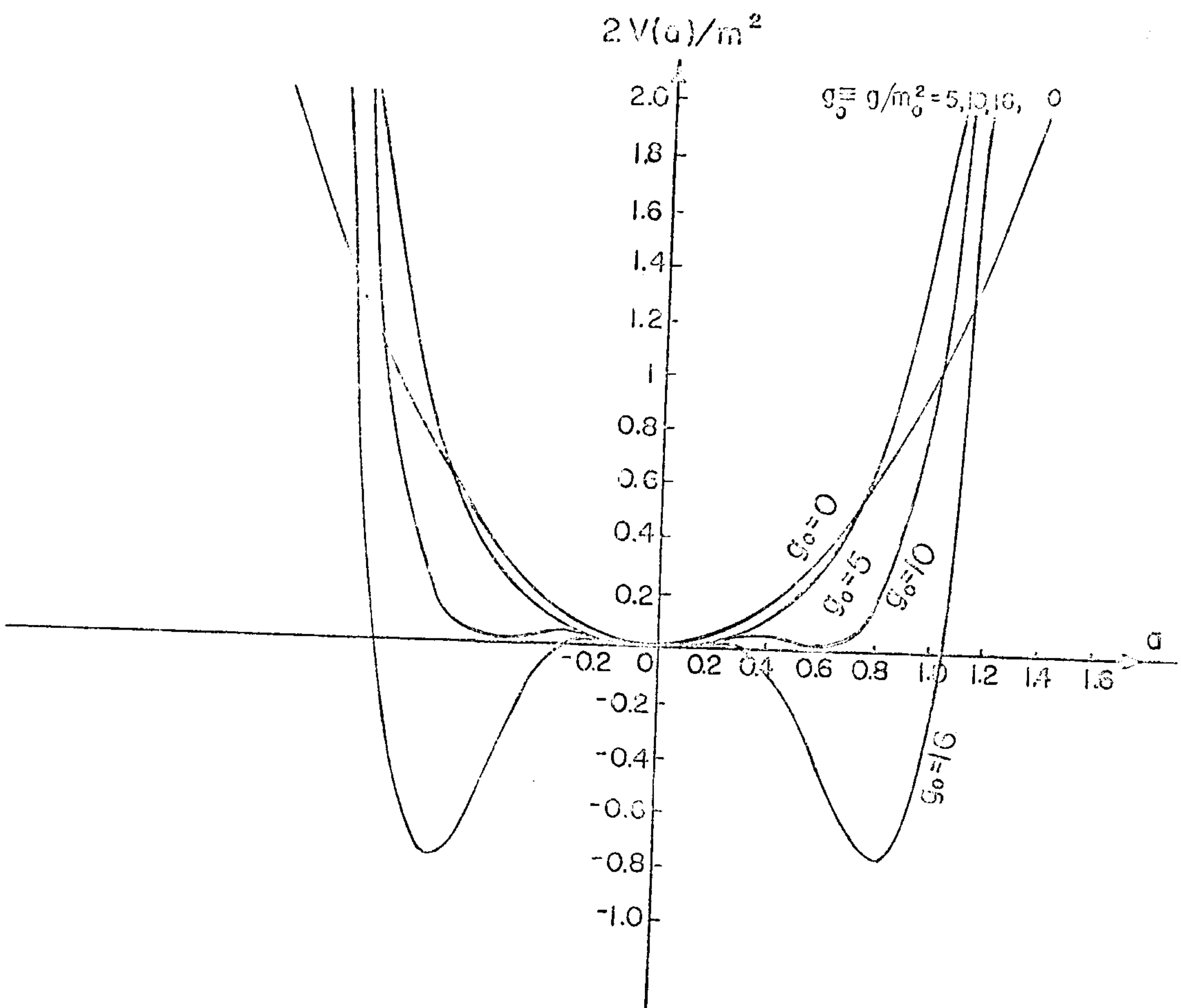


Fig. 3C



2V(a)/m²
g₀ ≡ g/m₀² = 5, 10, 16, 0
g₀=0
g₀=5
g₀=10
g₀=16
a
2.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1.0
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6

Fig. 4

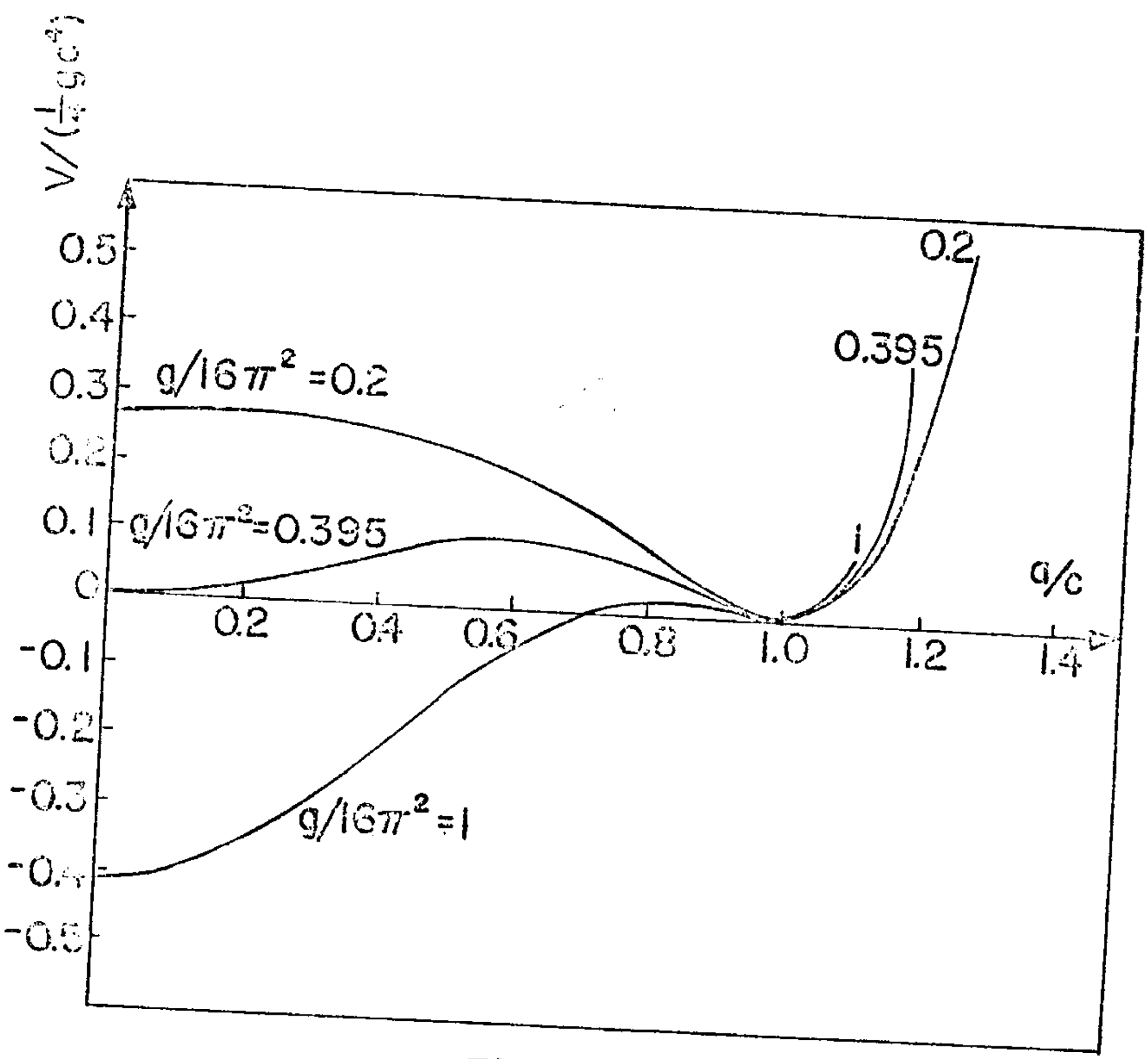

V/(1/4 g c^4)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
g/16π² =0.2
g/16π²=0.395
g/16π² =1
0.2
0.395
1
a/c
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Fig. 5